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Investigating Methods of Partial Fraction Decomposition

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Investigating Methods of Partial Fraction Decomposition

by

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Report

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Dedication

This dedication is first and foremost to my parents, David and Sharon Newberry, who always valued education from my earliest years. Thanks for believing in me. Thanks for supporting me both financially and with your words of encouragement. Thanks to my siblings, Aaron, Andrew, and Rhonda, for all of their belief in me and for helping me. Thanks to Aaron, who is one of the intelligent and best men I know. Thanks to Andy, who from an early age helped me develop a love for mathematics. Thanks to one of the best teachers in the state, Kathi Cook, for forcing me to stretch mathematically while in high school, and always having a level of teaching to strive for. I hope to someday think I am as good as you were. Thanks to all the students who have been on my mathematics teams over the years, who always pushed me to pursue higher mathematics. Without all of you, I would never have come this far. To my best friend, Robert LaChance, thanks for challenging me every time I talk to you. Thanks for trying to make me a better teacher and person. Thanks to Shri Ganeshram, Nathan Shih, Brett Hallum, Andrew Darling, Brian Gregg, Clayton Poteat, and Nick Smith, who had to deal with a stressed, overworked, and tired math coach, but who still made me a state champion math coach despite my lack of focus the past two years. Thanks to Christina Hoffmaster, who made me a better teacher by watching her teach for 11 years. Thanks to all in my math cohort, for being more than colleagues and classmates. Thanks for being my friends and causing me to grow. Thanks to all my professors for not only teaching me, but changing my teaching. Thanks for encouraging me to push kids higher, and think deeper mathematically.

Abstract

Investigating Methods of Partial Fraction Decomposition

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This report discusses the history of partial fractions. The paper investigates several methods of solving partial fraction decomposition problems. First, the basic method taught in most calculus courses is addressed. Another method addressed is a substitution method in which a clever substitution is used to transform expressions into something more easily integrable. Then, a rationale for partial fraction decomposition is given that includes a number theory perspective, a calculus perspective, and an algebraic perspective. A fourth method discussed involves using derivatives.

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Chapter 1

An Introduction

Partial fraction decomposition is a procedure used to reduce the degree of either the numerator or the denominator of a rational function and is often used to help in the computation of derivatives, antiderivatives, integrals, linear functional transformations [**2**, p. 552]. There are a few basic methods of using partial fraction decomposition utilized in most calculus courses, but there are many methods that will be addressed in this paper that would expedite the process. The method of partial fractions for solving integrals was introduced by John Bernoulli. Bernoulli was a Swiss mathematician who lived from 1667 to 1748, and was instrumental in the early development of calculus and taught many outstanding students, including Leonhard Euler. First, the basic methods of using partial fractions presented in calculus will be investigated, followed by the use of a method involving substitution. Also discussed will be the rationale for partial fraction decomposition, as well as a method of partial fraction decomposition using derivatives.

Chapter 2

Basic Method

It is likely that the first time many students encounter a problem using partial fraction decomposition will be in a high school or university Calculus course. Partial fraction decomposition will be taught as an integration technique to allow many integrals to be done by inspection, which could not be done efficiently without the use of partial fractions.

One example would be

$$\int \frac{3x+5}{(x+3)(x-1)}\,dx.$$

Using partial fractions, one would write an equation such as

$$\frac{3x+5}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}.$$

Then, one can rewrite the right side as

$$\frac{A(x-1)+B(x+3)}{(x+3)(x-1)}.$$

This yields the equation

$$\frac{3x+5}{(x+3)(x-1)} = \frac{Ax-A+Bx+3B}{(x+3)(x-1)}.$$

So, since the denominators of the two sides are equal, then the following system of equations is produced

$$A+B=3$$
$$5=-A+3B.$$

Thus, B = 2 and A = 1, so the fraction

$$\frac{3x+5}{(x+3)(x-1)}=\frac{1}{x+3}+\frac{2}{x-1},$$

which makes

$$\int\frac{3x+5}{(x+3)(x-1)}dx$$

transform into

$$\int\left(\frac{1}{x+3}+\frac{2}{x-1}\right)dx.$$

This equation is easily integrable by inspection,

$$\ln|x+3|+2\ln|x-1|+C.$$

Repeated Factors

When there are repeated linear factors, the common method for performing partial fraction decomposition is as follows:

$$\int \frac{x^2+3x-4}{x^3-4x^2+4x}\,dx = \int \frac{x^2+3x-4}{x\left(x^2-4x+4\right)}\,dx = \int \frac{x^2+3x-4}{x\left(x-2\right)^2}\,dx\,.$$

Then, one can write

$$\frac{x^2+3x-4}{x\left(x-2\right)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{\left(x-2\right)^2}\,,$$

and then by multiplying through by the common denominator $x\left(x-2\right)^2$, the equation becomes

$$x^2+3x-4 = A\left(x-2\right)^2 + Bx\left(x-2\right) + Cx\,.$$

Next one can expand and set the corresponding parts equal to each other, which yields the following system of equations

$$A+B=1,$$

$$-4A-2B+C=3,$$

$$4A=-4\,.$$

The solution to this system of equations is $A=-1$, $B=2$, $C=3$, and thus the original integral is transformed into an integral which can be more easily done.

$$\int \frac{x^2+3x-4}{x^3-4x^2+4x}\,dx = \int \left(\frac{-1}{x} + \frac{2}{x-2} + \frac{3}{\left(x-2\right)^2}\right)dx$$

Once again, this is easily integrable,

$$-\ln|x|+2\ln|x-2|-\frac{3}{x-2}+C.$$

Some partial fraction decomposition problems have a quadratic factor and a linear factor such as:

$$\int\frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}dx.$$

This fraction can be rewritten in the following manner:

$$\frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}=\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+4}.$$

Then, multiplying through by the common denominator of $x(x-1)\left(x^2+4\right)$ yields:

$$2x^3-4x-8=A(x-1)\left(x^2+4\right)+Bx\left(x^2+4\right)+(Cx+D)(x)(x-1)$$

Now, some new method is needed to make this process more efficient. Letting $x=0$,

$$-8=A(-1)(4)+0+0,$$

and thus $A=2$. Letting $x=1$, yields the following

$$-10=A(0)(5)+B(1)(5)+(C+D)(1)(0)$$

and thus $B=-2$.

Finding C and D in the equation is a little more involved as it involves letting x be two different values, which will create two equations for C and D, and then one can solve a system of equations. Letting $x=3$ will result in

$$34 = 2(2)(13) - 2(3)(13) + (3C + D)(3)(2)$$
$$10 = 3C + D.$$

Similarly, one can let $x=-2$ and the following result will occur

$$-16 = 2(-3)(8) - 2(-2)(8) + (-2C + D)(-2)(-3)$$
$$D = 2C.$$

Substitution will result in finding that $C=2$ and $D=4$. So,

$$\int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)}\,dx = \int \left(\frac{2}{x} - \frac{2}{x-1} + \frac{2x+4}{x^2+4}\right)dx = \int \left(\frac{2}{x} - \frac{2}{x-1} + \frac{2x}{x^2+4} + \frac{4}{x^2+4}\right)dx$$
$$= 2\ln|x| - 2\ln|x-1| + \ln\left(x^2+4\right) + 2\tan^{-1}\left(\frac{x}{2}\right) + C.$$

Without partial fractions, many integrals would either be impossible to perform or would be really difficult. Partial fraction decomposition is a skill which helps students be able to easily integrate.

Chapter 3

Substitution

The method of partial fraction decomposition of a rational function usually involves solving a system of linear equations. There is a quick method for finding the partial fraction decomposition of a rational function in the special case when the denominator is a power of a single linear or irreducible quadratic factor, so when the denominator is in the form $(ax+b)^k$ or $(ax^2+bx+c)^k$ with $4ac > b^2$ [**4**, p. 145].

This method provides a shortcut to the normal process for deriving the partial fraction decomposition. Using the example of

$$\frac{x^2+4x-3}{(x-2)^3},$$

one can use the substitution $t = x-2$ or more conveniently $x = t+2$. So, $\frac{x^2+4x-3}{(x-2)^3}$ becomes $\frac{t^2+8t+9}{t^3}$ through the following algebraic steps.

$$x^2+4x-3 = (t+2)^2+4(t+2)-3 = t^2+8t+9$$
$$(x-2)^3 = t^3.$$

This allows one to use the fraction $\frac{t^2+8t+9}{t^3}$, which can be split into

$$\frac{1}{t}+\frac{8}{t^2}+\frac{9}{t^3}.$$

From this fraction, using the substitution, $t = x-2$, it follows that the original fraction

$$\frac{x^2+4x-3}{(x-2)^3}=\frac{1}{x-2}+\frac{8}{(x-2)^2}+\frac{9}{(x-2)^3}.$$

The numbers 9, 8, and 1, in the numerators of the decomposition could also have been obtained as the remainders by successive division of x^2+4x-3 by $x-2$. [**4**, p. 145] This method also works for improper fractions, which eliminates the need for the initial polynomial division. This can be done if one assumes the denominator is *monic*, that is the coefficient of x term is $a=1$, and consider the rational function

$$R(x)=\frac{N(x)}{D(x)}$$

in the linear and irreducible quadratic cases separately.

First, one can show the linear case. Let $D(x)=(x+b)^k$. Let $x=t-b$. Then

$$\frac{N(x)}{(x+b)^k}=\frac{G(t)}{t^k}$$

and thus, this would immediately yield the desired decomposition [**4**, p. 146]. The coefficients of G are the coefficients in the numerators of the partial fractions and can be obtained by binomial expansion, as shown earlier, or as Taylor polynomial coefficients.

To illustrate the method, use the function

$$\frac{x^4+2x^3-x^2+5}{(2x-1)^5}. \qquad [\mathbf{4}, \text{p. } 146]$$

After factoring out the 2 and letting $x=t+\frac{1}{2}$, straightforward algebra converts this to

$$\frac{1}{32}\left(\frac{1}{t}+\frac{4}{t^2}+\frac{\frac{7}{2}}{t^3}+\frac{1}{t^4}+\frac{\frac{81}{16}}{t^5}\right),$$

so the decomposition is

$$\frac{\frac{1}{16}}{2x-1}+\frac{\frac{1}{2}}{(2x-1)^2}+\frac{\frac{7}{8}}{(2x-1)^3}+\frac{\frac{1}{2}}{(2x-1)^4}+\frac{\frac{81}{16}}{(2x-1)^5}.$$

One can perform similar computations when the power of the numerator is larger than the power of the denominator, considered an improper fraction. This can be seen in the example

$$\frac{3x^4-x^3+x^2-5}{(x-2)^3}.$$

Let $t=x-2$ and thus $x=t+2$, so $\frac{3x^4-x^3+x^2-5}{(x-2)^3}$ can be transformed into the following:

$$\frac{3(t+2)^4-(t+2)^3+(t+2)^2-5}{t^3}=\frac{3t^4+23t^3+67t^2+88t+39}{t^3}=3t+23+\frac{67}{t}+\frac{88}{t^2}+\frac{39}{t^3},$$

One can go further in attacking the irreducible quadratic case,

$$D(x)=\left(x^2+bx+c\right)^k$$

First, complete the square in order to express $D(x)$ in the form $\left[(x+p)^2+q\right]^k$. Then, make two substitutions, first $t=x+p$ as before, and then $s=t^2+q$. This is illustrated in the following example:

$$R(x)=\frac{4x^5-17x^4+45x^3-58x^2+48x-8}{\left(x^2-2x+3\right)^3}.$$

Then, after completing the square, the following is produced:

$$R(x)=\frac{4x^5-17x^4+45x^3-58x^2+48x-8}{\left[(x-1)^2+2\right]^3}.$$

Setting $t=x-1$ and substituting and simplifying yields the following expression:

$$\frac{4t^5+3t^4+17t^3+15t^2+19t+14}{\left(t^2+2\right)^3}.$$

Letting $s=t^2+2$ eventually results in

$$\frac{4t+3}{s}+\frac{t+3}{s^2}+\frac{t-4}{s^3},$$

which yields the decomposition

$$R(x)=\frac{4x-1}{x^2-2x+3}+\frac{x+2}{\left(x^2-2x+3\right)^2}+\frac{x-5}{\left(x^2-2x+3\right)^3}. \qquad [\mathbf{4}, \text{p. } 147]$$

Substitution allows one to change complex looking fractions to decompose into simpler fractions using clever changes.

Chapter 4

Rationale for Partial Fraction Decomposition and Other Methods

In Calculus, most students work with partial fractions, but few really understand why rational fractions can be written in terms of partial fractions. The rationale comes from abstract algebra, but a brief foray into the natural numbers can reveal much of the reasoning behind the use of partial fractions. One can take a journey through partial fractions by first looking at a general theorem stating the existence of partial fractions in calculus. Then, one can proceed by investigating the corresponding version in the natural numbers, and finish by discussing the calculus version and a generalization of the theorem to rational functions in one variable over an arbitrary field.

The purpose of this is to provide reasoning behind why partial fractions can be used. There are irreducible polynomials that rational expressions can be decomposed into and those irreducible polynomials are unique. The reasoning behind partial fractions is approached using a number theory perspective, a calculus perspective, and an algebraic perspective. First, a theorem involving rational functions is given. Every rational function $\frac{f(x)}{g(x)}$ of polynomials with real number coefficients can be written

$$\frac{f(x)}{g(x)} = \frac{f(x)}{q_1(x)^{a_1}\cdots q_n(x)^{a_n}}$$

$$= p_0(x) + \frac{p_{11}(x)}{q_1(x)} + \frac{p_{12}(x)}{\left(q_1(x)\right)^2} + \cdots + \frac{p_{1a_1}(x)}{\left(q_1(x)\right)^{a_1}} + \cdots + \frac{p_{n1}(x)}{q_n(x)} + \frac{p_{n2}(x)}{\left(q_n(x)\right)^2} + \cdots + \frac{p_{na_n}(x)}{\left(q_n(x)\right)^{a_n}}$$

where each $q_i(x)$ is an irreducible polynomial and each $p_{ij}(x)$ is a polynomial whose degree is less than that of $q_i(x)$. Furthermore, $p_0(x)$ and $p_{ij}(x)$ are unique [5, p. 362]. Before investigating this with polynomials, one can look at rational numbers.

In other words, one can show every rational number can be written as a sum of proper fractions, where the denominators are all primes or positive integer powers of primes. Question: Do integers *a* and *b* exist such that

$$\frac{1}{28} = \frac{a}{4} + \frac{b}{7} = \frac{a}{2^2} + \frac{b}{7}? \qquad [\mathbf{5}, \text{p. } 363]$$

Yes, there exists many solutions, one of which is $a = -1$ and $b = 2$. One can multiply through by 28 to obtain the equation

$$1 = 7a + 4b,$$

and this has integer solutions because 4 and 7 are *relatively prime* [**5**, p. 363]. Two integers are relatively prime, if they have no integral divisors in common besides 1. Furthermore, is there a natural and non-trivial way to find *e*, *f*, and *g* so that

$$\frac{5}{8} = \frac{e}{2} + \frac{f}{2^2} + \frac{g}{2^3}? \qquad [\mathbf{2}, \text{p.}363]$$

Multiplying through by 8 to find

$$5 = e \cdot 2^2 + f \cdot 2 + g,$$

suggests taking *e*, *f*, and *g* as digits in the base 2 expansion of 5, which yields the result $e = 1, f = 0, g = 1$. Since expansion in any base is unique this method provides an appealing algorithm for resolving this question. This method doesn't just work for base 2 expansions. This method can be extended to work for any denominator, p^r, where *p* is a prime. [**5**, p.364]

Next, one can explore partial fractions in calculus with a little twist, because we want to explore the reasoning underlying the process of partial fractions, rather than just seeking the most efficient solution. Looking at an example, points to the reasoning behind wanting to use long division first when working these types of problems. Using long division

$$\frac{x^6+24x^5+70x^4+79x^3+32x^2+11x-9}{x^6+2x^5+x^4-x^2-2x-1},$$

can be rewritten as

$$1+\frac{22x^5+69x^4+79x^2+33x^2+13x-8}{x^6+2x^5+x^4-x^2-2x-1}\ .\ [\mathbf{5},\ \text{p.365}]$$

Using synthetic division, one can factor the denominator into powers of irreducible polynomials to achieve

$$x^6+2x^5+x^4-x^2-2x-1=(x+1)^3(x^2+1)(x-1),$$

and thus

$$1+\frac{22x^5+69x^4+79x^2+33x^2+13x-8}{x^6+2x^5+x^4-x^2-2x-1}=1+\frac{22x^5+69x^4+79x^2+33x^2+13x-8}{(x+1)^3(x^2+1)(x-1)}.$$

This can be rewritten in the form

$$1+\frac{2x^2+7x+10}{(x+1)^3}+\frac{7x+11}{x^2+1}+\frac{13}{x-1}$$

by using basic techniques for partial fraction decomposition and solving a system of equations using matrices by way of reduced echelon form.

At this point $\frac{2x^2+7x+10}{(x+1)^3}$ needs to be decomposed. Since

$$2x^2+7x+10=(x+1)(2x+5)+5=(x+1)\big(2(x+1)+3\big)+5=2(x+1)^2+3(x+1)+5,$$

then

$$\frac{2x^2+7x+10}{(x+1)^3}=\frac{2(x+1)^2}{(x+1)^3}+\frac{3(x+1)}{(x+1)^3}+\frac{5}{(x+1)^3}=\frac{2}{x+1}+\frac{3}{(x+1)^2}+\frac{5}{(x+1)^3},$$

so polynomials can be rewritten and broken down much as numerical fractions can. This resembles the numerical fraction and how it was broken down into a base problem. One could have also used the substitution method addressed previously in this paper so the following change can be made by letting $x=t-1$.

$$\frac{2x^2+7x+10}{(x+1)^3}=\frac{2(t-1)^2+7(t-1)+10}{t^3}=\frac{2t^2+3t+5}{t^3}=\frac{2}{t}+\frac{3}{t^2}+\frac{5}{t^3}$$

Substituting back in for t yields

$$\frac{2}{x+1}+\frac{3}{(x+1)^2}+\frac{5}{(x+1)^3}.$$

Thus,

$$1+\frac{2x^2+7x+10}{\left(x+1\right)^3}+\frac{7x+11}{x^2+1}+\frac{13}{x-1}=1+\frac{2}{x+1}+\frac{3}{\left(x+1\right)^2}+\frac{5}{\left(x+1\right)^3}+\frac{7x+11}{x^2+1}+\frac{13}{x-1}.$$

[**5**, p. 365].

There is a question of whether dividing by the denominator is necessary or not. The answer is that doing so results in a unique factorization of polynomials and give an example what will happen if one doesn't divide first. If one doesn't divide first, then some nonunique answers can occur. What can happen is that multiple solutions can be found because of a problem of the numerator not having a lower power than the denominator. Dividing first solves this issue, providing for a unique decomposition. This method of partial fraction decomposition results in many terms that can be easily integrated [**5**, p. 365] .

The denominator can always be factored into powers of irreducibles because irreducibles play the role of the primes in the set of integers. Just as every integer can be uniquely written as the product of powers of primes, every polynomial can be uniquely written as the product of powers of irreducible polynomials. This property is called unique factorization. Since each polynomial in one variable can be written as a unique factorization, we have a notion of relatively prime polynomials. A couple of key results happen because of relatively prime polynomials. First, the polynomials have no common irreducible factors, and furthermore $f\left(x\right)$ and $g\left(x\right)$ are relatively prime if there exist polynomials $m\left(x\right)$ and $n\left(x\right)$ so that

$$f(x)m\left(x\right)+g(x)n\left(x\right)=1.$$

Why couldn't one stop at

$$\frac{2x^2+7x+10}{\left(x+1\right)^3}?$$

The first reason is that

$$\frac{2x^2+7x+10}{(x+1)^3}$$

is not yet easy to integrate. The second reason is that further decomposition of a rational function results from rewriting the numerator using the polynomial in the denominator as the base in a way parallel to the base decomposition in the number theory part of this paper. Polynomials are automatically written in base *x*; the coefficients encode the number of each power of *x*. However, one could write a polynomial in any other base, just as was done with numbers, writing numbers in base 10, base 2, base 16. In this case, $2x^2+7x+10$ was written in the base $x+1$ [**5**, p. 365].

One can see how the decomposition of polynomials compares to the decomposition of numbers into bases. Thus, one can discuss the existence of the partial fraction decomposition due to the existence of the Euclidean algorithm over the real numbers. Suppose that $\frac{p(x)}{q(x)}$ can be written as a partial fraction decomposition in two ways:

$$\frac{p(x)}{q(x)}=\frac{r(x)}{s(x)}+\frac{t(x)}{v(x)}=\frac{R(x)}{s(x)}+\frac{T(x)}{v(x)}, \qquad \text{[5, p. 367]}$$

where all fractions are proper and reduced, $q(x)=s(x)v(x)$, and $s(x)$ and $v(x)$ have no nonconstant common factors. Then, by multiplying through by $s(x)v(x)$, one can obtain

$$p(x)=r(x)v(x)+s(x)t(x)=R(x)v(x)+s(x)T(x).$$

Hence,

$$s(x)\big(t(x)-T(x)\big)=v(x)\big(R(x)-r(x)\big),$$

which implies that $v(x) \mid t(x)-T(x)$. Since both $t(x)$ and $T(x)$ have lower degrees than $v(x)$, then the only way $v(x)$ divides their difference is if $t(x)-T(x)=0$. By a similar argument, it can be shown that $r(x)-R(x)=0$. This allows one to state that $T(x)=t(x)$ and $R(x)=r(x)$. Thus proving partial fraction decomposition has uniqueness when the denominators are relatively prime. Finally, notice that expansion in any polynomial base is unique, since the division algorithm yields a unique quotient and remainder. [**5**, p. 367] Therefore, $p_0(x)$ and $p_{ij}(x)$ are unique, which was to be proven.

Chapter 5

Using $f'(x)$ in Partial Fraction Decomposition

In most Calculus textbooks, the partial fraction decomposition for an irreducible quadratic factor ax^2+bx+c involves a term of the form $\dfrac{Bx+C}{ax^2+bx+c}$. However, unless $b=0$, students will still need to split this expression up further before integrating. This can be seen in the following example,

$$\frac{2}{x\left(x^2+2x+2\right)}=\frac{A}{x}+\frac{Bx+C}{\left(x^2+2x+2\right)}=\frac{1}{x}+\frac{-x-2}{\left(x^2+2x+2\right)}. \qquad [\mathbf{3}, \text{p. } 221]$$

Then, the last part of $\dfrac{-x-2}{\left(x^2+2x+2\right)}$ must be broken down further into

$$-\frac{x+1}{x^2+2x+2}-\frac{1}{x^2+2x+2}=-\frac{x+1}{x^2+2x+2}-\frac{1}{(x+1)^2+1}$$

before integrating. This second step often confuses students, and many textbooks just avoid it by always having $b=0$ in all exercises [**3**, p.221]. By changing the order of computation and using the procedure shown below, the process will be easier to navigate.

For each factor in the denominator, linear or quadratic, let one of the partial fractions be

$$\frac{Af'(x)}{f(x)},$$

and if $f(x)$ is quadratic, add the term

$$\frac{B}{f(x)}.$$

For $\left[f(x)\right]^2$, also add in

$$\frac{Cf'(x)}{\left[f(x)\right]^2},$$

and also

$$\frac{D}{\left[f(x)\right]^2},$$

again if $f(x)$ is quadratic. For $\left[f(x)\right]^3$, the pattern would continue. The number of terms used is equal to the degree of the original denominator.

Consider an example such as $\dfrac{2}{x\left(x^2+2x+2\right)}$. Think of x as $f(x)$ and $\left(x^2+2x+2\right)$ as $g(x)$. Then, $f'(x)=1$ and $g'(x)=\left(2x+2\right)$. So,

$$\frac{2}{x\left(x^2+2x+2\right)}$$

can be rewritten as

$$A\frac{f'(x)}{f(x)}+B\frac{g'(x)}{g(x)}+C\frac{1}{g(x)}=A\frac{1}{x}+B\frac{2x+2}{x^2+2x+2}+C\frac{1}{x^2+2x+2}.$$

Multiplying through by the least common denominator $f(x)g(x)$ yields

$$A\left(x^2+2x+2\right)+Bx\left(2x+2\right)+Cx=2.$$

Therefore,

$$\begin{aligned}&A+2B=0,\\&2A+2B+C=0,\\&2A=2.\end{aligned}$$

Solving the system, one will find that $A=1$, $B=-\frac{1}{2}$, and $C=-1$, and substituting in for A, B, and C:

$$A\frac{1}{x}+B\frac{2x+2}{x^2+2x+2}+C\frac{1}{x^2+2x+2}=\frac{1}{x}+\left(-\frac{1}{2}\right)\frac{2x+2}{x^2+2x+2}-\frac{1}{x^2+2x+2}$$

This makes it easier to do the integral by inspection

$$\int\left[\frac{1}{x}+\left(-\frac{1}{2}\right)\frac{2x+2}{x^2+2x+2}-\frac{1}{x^2+2x+2}\right]dx=\int\left[\frac{1}{x}+\left(-\frac{1}{2}\right)\frac{2x+2}{x^2+2x+2}-\frac{1}{\left(x+1\right)^2+1}\right]dx$$

$$=\ln|x|-\frac{1}{2}\ln\left(x^2+2x+2\right)-\tan^{-1}\left(x+1\right)+C.$$

There are just as many coefficients to be found using the second method, but the advantage is that no further decomposition is needed. The expression is ready to be integrated as opposed to the prior method, which was shown [**3**, p. 221].

The rationale behind this method can be found by considering an example where one has a fraction in the form $\frac{K}{f(x)g(x)}$, where K is a constant, and $f(x)$, $g(x)$ are functions such that $f(x)$ is linear and $g(x)$ is quadratic. Thus, $f'(x)$ is a constant and $g'(x)$ is linear. Notice that $\frac{K}{f(x)g(x)}$ can be rewritten as a partial fraction in the form

$$\frac{K}{f(x)g(x)}=\frac{A_1}{f(x)}+\frac{B_1x+C_1}{g(x)},$$

where A_1, B_1, and C_1 are constants.

Using the derivative method discussed in this chapter, one can rewrite $\frac{K}{f(x)g(x)}$ in the form

$$\frac{K}{f(x)g(x)}=A_2\frac{f'(x)}{f(x)}+B_2\frac{g'(x)}{g(x)}+C_2\frac{1}{g(x)},$$

where A_2, B_2, and C_2 are constants. But, since $f'(x)$ is constant, then $A_2f'(x)$ is constant. And, since $g'(x)$ is linear, then $B_2g'(x)$ is linear, and when added to C_2, this will yield a linear equation. So $A_2\frac{f'(x)}{f(x)}+B_2\frac{g'(x)}{g(x)}+C_2\frac{1}{g(x)}$, will produce a constant divided by $f(x)$ summed with a linear function divided by $g(x)$, and thus

$$\frac{K}{f(x)g(x)} = A_2 \frac{f'(x)}{f(x)} + B_2 \frac{g'(x)}{g(x)} + C_2 \frac{1}{g(x)}$$

produces an equivalent result that could have been produced by

$$\frac{K}{f(x)g(x)} = \frac{A_1}{f(x)} + \frac{B_1 x + C_1}{g(x)}.$$

This method again allows one to make the partial fraction easier to break apart and increases the efficiency because it prevents from having to break up the second fraction further.

Chapter 7

Conclusion

Partial fraction decomposition can be a very cumbersome task. Exploring alternative decomposition methods that expedite the process can be quite helpful in making partial fraction decomposition not seem so daunting. Understanding the basic concepts and learning some shortcuts and knowing how to recognize when to use these shortcuts is essential to being competent, confident, and efficient at solving partial fraction decomposition problems. Using clever substitutions, division, and derivatives to help find partial fraction decompositions can turn some complicated problems into simpler problems. An understanding of the methods and the rationale or proof behind some of these methods has been enlightening.

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Vita

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